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Linear Algebra and its Applications 403 (2005) 1–23

www.elsevier.com/locate/laa

A refinement of the split decomposition of a tridiagonal pair

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Received 18 October 2004; accepted 13 December 2004
Available online 8 April 2005
Submitted by R.A. Brualdi

Abstract

Let V denote a nonzero finite dimensional vector space over a field $\mathbb{K}$, and let (A, A^*) denote a tridiagonal pair on V of diameter d. Let $V = U_0 + \cdots + U_d$ denote the split decomposition, and let ρ_i denote the dimension of U_i. In this paper, at first we show there exists a unique integer h $(0 \leqslant h \leqslant d/2)$ such that $\rho_{i-1} < \rho_i$ for $1 \leqslant i \leqslant h$, $\rho_{i-1} = \rho_i$ for $h < i \leqslant d-h$ and $\rho_{i-1} > \rho_i$ for $d-h < i \leqslant d$. We call h the *height* of the tridiagonal pair. For $0 \leqslant r \leqslant h$, we define subspaces $U_i^{(r)}$ $(r \leqslant i \leqslant d-r)$ by $U_i^{(r)} = R^{i-r}(U_r \cap \operatorname{Ker} R^{d-2r+1})$, where R denotes the rasing map. We show V is decomposed as a direct sum $V = \sum_{r=0}^{h} \sum_{i=r}^{d-r} U_i^{(r)}$. This gives a refinement of the split decomposition. Define $U^{(r)} = \sum_{i=r}^{d-r} U_i^{(r)}$, and observe $V = \sum_{r=0}^{h} U^{(r)}$. We show $LU^{(r)} \subseteq U^{(r-1)} + U^{(r)} + U^{(r+1)}$ for $0 \leqslant r \leqslant h$, where we set $U^{(-1)} = U^{(h+1)} = 0$. Let $F^{(r)} : V \to U^{(r)}$ denote the projection. We show the lowering map L is decomposed as $L = L^{(-)} + L^{(0)} + L^{(+)}$, where $L^{(-)} = \sum_{r=1}^{h} F^{(r-1)} L F^{(r)}$, $L^{(0)} = \sum_{r=0}^{h} F^{(r)} L F^{(r)}$, and $L^{(+)} = \sum_{r=0}^{h-1} F^{(r+1)} L F^{(r)}$. These maps satisfy $L^{(-)} U^{(r)} \subset U^{(r-1)}$, $L^{(0)} U^{(r)} \subseteq U^{(r)}$, and $L^{(+)} U^{(r)} \subseteq U^{(r+1)}$ for $0 \leqslant r \leqslant h$. The main results of this paper are the following: (i) For $0 \leqslant r \leqslant h-1$ and $r+2 \leqslant i \leqslant d-r-1$, $RL^{(+)} = \alpha L^{(+)} R$ holds on $U_i^{(r)}$ for some scalar α; (ii) For $1 \leqslant r \leqslant h$ and $r \leqslant i \leqslant d-r-1$, $RL^{(-)} = \beta L^{(-)} R$ holds on $U_i^{(r)}$ for some scalar β; (iii) For $0 \leqslant r \leqslant h$ and

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doi:10.1016/j.laa.2004.12.034

$r+1 \leqslant i \leqslant d-r-1$, $RL^{(0)} = \beta L^{(0)} R + \gamma I$ holds on $U_i^{(r)}$ for some scalars γ, δ. Moreover we give explicit expressions of α, β, γ, δ.

AMS classification: 05E30; 05E35; 33C45; 33D45

Keywords: Askey–Wilson relation; Tridiagonal pair; Leonard pair

1. Introduction

The notion of a tridiagonal pair was introduced by Ito et al. [1], generalizing the notion of a Leonard pair which had been introduced by Terwilliger [2]. See Terwilliger's survey [3] about Leonard pairs and tridiagonal pairs.

A tridiagonal pair is defined as follows.

Definition 1.1 [1]. Let V denote a nonzero finite dimensional vector space over a field $\mathbb{K}$. By a *tridiagonal pair* on V, we mean a pair (A, A^*), where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy the following conditions:

(i) A and A^* are both diagonalizable on V.
(ii) There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leqslant i \leqslant d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$.
(iii) There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leqslant i \leqslant \delta),$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.
(iv) There is no subspace W of V such that both $AW \subseteq W$, $A^*W \subseteq W$, other than $W = 0$ and $W = V$.

Remark 1.2. With reference to Definition 1.1, it is known that $d = \delta$ [1, Corollary 5.7]. We call this common value the *diameter* of the pair.

Throughout this paper, we fix the following notation. Let $\mathbb{K}$ denote a field and let V denote a nonzero finite dimensional vector space over $\mathbb{K}$. Let (A, A^*) denote a tridiagonal pair on V with diameter d. Let $V_0, V_1, \ldots, V_d$ (respectively $V_0^*, V_1^*, \ldots, V_d^*$) denote an ordering of the eigenspaces of A (respectively A^*) that satisfies the condition (ii) (respectively (iii)) in Definition 1.1. Let θ_i (respectively θ_i^*) denote the eigenvalue of A (respectively A^*) for the eigenspace V_i (respectively V_i^*), and let ρ_i denote the dimension of V_i for $0 \leqslant i \leqslant d$.

It is known that $\rho_i = \rho_{d-i}$ and $\rho_i \leqslant \rho_{i+1}$ for $0 \leqslant i < d/2$ [1, Corollay 6.6], so that the sequence $\rho_0, \rho_1, \ldots, \rho_d$ is unimodal. In Section 3, we show that there exists a unique integer $h (0 \leqslant h \leqslant d/2)$ such that

$$\rho_{i-1} < \rho_i \quad (1 \leqslant i \leqslant h), \qquad \rho_{i-1} = \rho_i \quad (h < i \leqslant d-h),$$
$$\rho_{i-1} > \rho_i \quad (d-h < i \leqslant d).$$

We call h the *height* of the tridiagonal pair.

In Section 2, we recall some basic facts about the tridiagonal pairs which are given in [1]. In particular, the space V is decomposed as a direct sum

$$V = U_0 + U_1 + \cdots + U_d,$$

and there are linear transformations $R, L : V \to V$ such that $RU_i \subseteq U_{i+1}$ and $LU_i \subseteq U_{i-1}$. The above decomposition is called the *split decomposition* of the tridiagonal pair.

In Sections 4 and 5, we introduce a direct sum decomposition

$$V = U^{(0)} + U^{(1)} + \cdots + U^{(h)},$$

where h denotes the height, having the following property:

$$RU^{(r)} \subseteq U^{(r)}, \qquad LU^{(r)} \subseteq U^{(r-1)} + U^{(r)} + U^{(r+1)}.$$

We show that V is decomposed as

$$V = \sum_{r=0}^{h} \sum_{i=r}^{d-r} U^{(r)} \cap U_i \qquad \text{(direct sum)}.$$

This gives a refinement of the split decomposition.

In Section 6, we show that the map L is decomposed as

$$L = L^{(-)} + L^{(0)} + L^{(+)},$$

and the three maps satisfy

$$L^{(-)}U^{(r)} \subseteq U^{(r-1)}, \qquad L^{(0)}U^{(r)} \subseteq U^{(r)}, \qquad L^{(+)}U^{(r)} \subseteq U^{(r+1)}.$$

We also show that each of $(R, L^{(-)})$, $(R, L^{(+)})$ satisfies the q-Serre relation.

In Sections 8–10, we study linear relations of $L^{(+)}$, $L^{(-)}$, $L^{(0)}$ with R. We show that $RL^{(+)}$ (respectively $RL^{(-)}$) is a scalar multiple of $L^{(+)}R$ (respectively $L^{(-)}R$) on each $U^{(r)} \cap U_i$. We also give a linear relation between $RL^{(0)}$, $L^{(0)}R$ and I (the identity map) on each $U^{(r)} \cap U_i$.

2. The split decomposition and the tridiagonal relation

In this section, we recall some known facts about the tridiagonal pairs, which were given in [1]. For $0 \leqslant i \leqslant d$, we set

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d). \tag{1}$$

Lemma 2.1 [1, Theorem 4.6]. *The space V is decomposed as*

$$V = U_0 + U_1 + \cdots + U_d \quad (\textit{direct sum}). \tag{2}$$

The decomposition given in (2) is called the *split decomposition* of the tridiagonal pair.

Lemma 2.2 [1, Corollary 5.7]. *For* $0 \leqslant i \leqslant d$,

$$\dim V_i = \dim V_i^* = \dim U_i = \rho_i.$$

Lemma 2.3 [1, Corollaries 5.7 and 6.6]. *For* $0 \leqslant i < d/2$,

$$\rho_i \leqslant \rho_{i+1}, \quad \rho_i = \rho_{d-i}.$$

Let $F_i : V \to U_i$ denote the projection with respect to the direct sum (2). Then for $0 \leqslant i, j \leqslant d$,

$$F_0 + F_1 + \cdots + F_d = I, \quad F_i F_i = F_i, \quad F_i F_j = 0 \text{ if } i \neq j. \tag{3}$$

The *raising map* R and the *lowering map* L are defined as follows:

$$R = A - \sum_{i=0}^{d} \theta_i F_i, \quad L = A^* - \sum_{i=0}^{d} \theta_i^* F_i. \tag{4}$$

Lemma 2.4 [1, Lemma 6.3]

(i) $RU_i \subseteq U_{i+1}$ $(0 \leqslant i \leqslant d-1)$, $RU_d = 0$.
(ii) $LU_i \subseteq U_{i-1}$ $(1 \leqslant i \leqslant d)$, $LU_0 = 0$.

Lemma 2.5 [1, Lemma 6.4]

(i) $RF_i = F_{i+1}R$ $(0 \leqslant i \leqslant d-1)$, $RF_d = 0$, $F_0R = 0$.
(ii) $LF_i = F_{i-1}L$ $(1 \leqslant i \leqslant d)$, $LF_0 = 0$, $F_dL = 0$.

Lemma 2.6 [1, Lemma 6.5]. *For* $0 \leqslant i < d/2$,

(i) *The restriction* $R^{j-i}|_{U_i} : U_i \to U_j$ *is an injection for* $i \leqslant j \leqslant d-i$.
(ii) *The restriction* $R^{d-2i}|_{U_i} : U_i \to U_{d-i}$ *is a bijection.*

Lemma 2.7. *Let W denote a subspace of V. Suppose that $RW \subseteq W$, $LW \subseteq W$ and $F_iW \subseteq W$ for $0 \leqslant i \leqslant d$. Then $W = 0$ or $W = V$.*

Proof. Observe that A and A^* are represented as linear combinations of R, L, F_i $(0 \leqslant i \leqslant d)$ by (4), so that $AW \subseteq W$ and $A^*W \subseteq W$. Now the result follows from Definition 1.1(iv). $\square$

Lemma 2.8 [1, Theorem 10.1]. *There is a sequence of scalars β, γ, ϱ taken from $\mathbb{K}$ such that*

$$\left[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*\right] = 0,$$

where $[B, C] = BC - CB$. The sequence is unique if the diameter is at least three.

The above relation is known as the *tridiagonal relation*. This implies a following relation between R and L. Let ε_i $(0 \leqslant i \leqslant d-2)$ denote the scalar defined by

$$\varepsilon_i = (\theta_i - \theta_{i+2})(\theta^*_{i+1} - \theta^*_{i+2}) - (\theta^*_{i+2} - \theta^*_i)(\theta_{i+1} - \theta_i). \tag{5}$$

Lemma 2.9 [1, Theorem 12.1]. *For* $0 \leqslant i \leqslant d-2$,

$$\left(R^3L - (\beta+1)R^2LR + (\beta+1)RLR^2 - LR^3 + (\beta+1)\varepsilon_i R^2\right)F_i = 0. \tag{6}$$

3. The height of a tridiagonal pair

Lemma 3.1. *Suppose $0 \leqslant i \leqslant d-2$, and let Y denote a subspace of U_i. If $RLY \subseteq Y$, $LRY \subseteq Y$ and $LR^2Y \subseteq RY$, then $LR^3Y \subseteq R^2Y$.*

Proof. Pick any vector u in Y. Applying (6) to u,

$$R^3Lu - (\beta+1)R^2LRu + (\beta+1)RLR^2u - LR^3u + (\beta+1)\varepsilon_i R^2u = 0.$$

Observe that each of $R^3Lu = R^2(RLu)$, $R^2LRu = R^2(LRu)$, $RLR^2u = R(LR^2u)$ and R^2u lies in R^2Y by our assumption. Hence LR^3u is in R^2Y. $\square$

Lemma 3.2. *Suppose $\rho_i = \rho_{i+1}$ for some i with $0 \leqslant i < d/2$. Then $\rho_i = \rho_j$ for all j with $i \leqslant j \leqslant d-i$.*

Proof. Define subspaces $W_0, W_1, \ldots, W_d$ by

$$W_k = \begin{cases} U_k & \text{if } 0 \leqslant k \leqslant i, \\ R^{k-i}U_i & \text{if } i+1 \leqslant k \leqslant d. \end{cases}$$

We set $W_{-1} = 0$. Observe that

$$RW_k \subseteq W_{k+1} \quad (0 \leqslant k \leqslant i-1),$$
$$RW_k = W_{k+1} \quad (i \leqslant k \leqslant d-1).$$

We show

$$LW_j \subseteq W_{j-1} \quad (0 \leqslant j \leqslant d) \tag{7}$$

by induction on j. Clearly (7) holds for $0 \leqslant j \leqslant i+1$. Observe that $RU_i = U_{i+1}$ by our assumption $\rho_i = \rho_{i+1}$ and Lemma 2.6, so we have $W_{i+1} = U_{i+1}$ and hence

(7) holds at $j = i + 2$. Now suppose $i + 3 \leqslant j \leqslant d$. Set $Y = W_{j-3}$ and observe that $RY = W_{j-2}$, $R^2Y = W_{j-1}$ and $R^3Y = W_j$. By induction,

$$RLY = RLW_{j-3} \subseteq RW_{j-4} \subseteq W_{j-3} = Y,$$
$$LRY = LRW_{j-3} = LW_{j-2} \subseteq W_{j-3} = Y,$$
$$LR^2Y = LW_{j-1} \subseteq W_{j-2} = RY.$$

Applying Lemma 3.1, we get $LR^3Y \subseteq R^2Y$, and this implies (7).

Now set $W = \sum_{k=0}^{d} W_k$. We have $RW \subseteq W$ and $F_\ell W = W$ $(0 \leqslant \ell \leqslant d)$. We have also $LW \subseteq W$ by (7). Therefore $W = V$ by Lemma 2.7. This implies $W_i = U_i$ for $0 \leqslant i \leqslant d$. In particular, $R^{j-i}U_i = U_j$ $(i \leqslant j \leqslant d - i)$, and this implies $\dim U_i = \dim U_j$ for $i \leqslant j \leqslant d - i$ by Lemma 2.6. □

Theorem 3.3. *There exists a unique integer h with $0 \leqslant h \leqslant d/2$ such that*

$$\rho_i < \rho_{i+1} \quad (0 \leqslant i \leqslant h - 1),$$
$$\rho_i = \rho_h \quad (h \leqslant i \leqslant d - h),$$
$$\rho_i > \rho_{i+1} \quad (d - h \leqslant i \leqslant d - 1).$$

Proof. Follows from Lemmas 3.2 and 2.3. □

By the *height* of the tridiagonal pair, we mean the integer h in Theorem 3.3. For the rest of this paper, let h denote the height of the tridiagonal pair.

4. The subspace $U_i^{(r)}$

For $0 \leqslant r \leqslant h$ and $r \leqslant i \leqslant d - r$, set

$$U_i^{(r)} = R^{i-r}(U_r \cap \operatorname{Ker} R^{d-2r+1}). \tag{8}$$

Lemma 4.1. *The following hold for $0 \leqslant r \leqslant h$.*

(i) $U_0^{(0)} = U_0$ *and* $U_d^{(0)} = U_d$.
(ii) $U_i^{(r)} \subseteq U_i$ $(r \leqslant i \leqslant d - r)$.
(iii) $U_r^{(r)} = U_r \cap \operatorname{Ker} R^{d-2r+1}$.
(iv) $U_i^{(r)} = R^{i-r}U_r^{(r)}$ $(r \leqslant i \leqslant d - r)$.
(v) $RU_i^{(r)} = U_{i+1}^{(r)}$ $(r \leqslant i \leqslant d - r - 1)$, $RU_{d-r}^{(r)} = 0$.
(vi) *The restriction* $R|_{U_i^{(r)}} : U_i^{(r)} \to U_{i+1}^{(r)}$ *is a bijection* $(r \leqslant i \leqslant d - r - 1)$.

Proof. Use Lemma 2.6 to get (vi) and $U_d^{(0)} = U_d$. The other assertions follow from (8). □

Lemma 4.2. *For* $1 \leqslant r \leqslant h$,

$$U_r = RU_{r-1} + U_r^{(r)} \quad (\textit{direct sum}).$$

Proof. Pick any vector v in $(RU_{r-1}) \cap U_r^{(r)}$. Then $Ru = v$ for some vector u in U_{r-1}. Observe that $R^{d-2r+2}u = R^{d-2r+1}v = 0$ since v is contained in $U_r^{(r)} \subseteq \operatorname{Ker} R^{d-2r+1}$. This implies $u = 0$ by Lemma 2.6, so that $v = 0$. Thus $(RU_{r-1}) \cap U_r^{(r)} = 0$.

Let N denote the restriction $R^{d-2r+1}|_{U_r}$. Observe $N : U_r \to U_{d-r+1}$ is a surjection, since $R^{d-2r+2}|_{U_{r-1}}$ is a bijection $U_{r-1} \to U_{d-r+1}$ by Lemma 2.6. So we get $\dim U_r = \dim U_{d-r+1} + \dim \operatorname{Ker} N$, and this implies $\rho_r = \rho_{d-r+1} + \dim \operatorname{Ker} N = \rho_{r-1} + \dim \operatorname{Ker} N$, here $\dim RU_{r-1} = \rho_{r-1}$ by Lemma 2.6. On the other hand, $\operatorname{Ker} N = U_r \cap \operatorname{Ker} R^{d-2r+1} = U_r^{(r)}$. Thus $\dim U_r = \dim RU_{r-1} + \dim U_r^{(r)}$. $\square$

Lemma 4.3. *For* $0 \leqslant r \leqslant h$,

$$\dim U_i^{(r)} = \rho_r - \rho_{r-1} \quad (r \leqslant i \leqslant d - r), \tag{9}$$

where we set $\rho_{-1} = 0$.

Proof. Follows from Lemmas 4.2 and 4.1(vi). $\square$

Lemma 4.4. *The subspace* $U_i^{(r)}$ *is nonzero for* $0 \leqslant r \leqslant h$ *and* $r \leqslant i \leqslant d - r$.

Proof. Follows from Theorem 3.3 and (9). $\square$

Lemma 4.5. *For* $1 \leqslant j \leqslant i \leqslant h$,

$$U_i = R^{i-j+1}U_{j-1} + \sum_{r=j}^{i} U_i^{(r)} \quad (\textit{direct sum}). \tag{10}$$

Proof. By Lemma 4.2, (10) holds at $j = i$. Suppose $1 \leqslant j < i$ and (10) holds at $j + 1$;

$$U_i = R^{i-j}U_j + \sum_{r=j+1}^{i} U_i^{(r)}.$$

Since $U_j = RU_{j-1} + U_j^{(j)}$ by Lemma 4.2,

$$\begin{aligned} U_i &= R^{i-j}\big(RU_{j-1} + U_j^{(j)}\big) + \sum_{r=j+1}^{i} U_i^{(r)} \\ &= R^{i-j+1}U_{j-1} + R^{i-j}U_j^{(j)} + \sum_{r=j+1}^{i} U_i^{(r)}. \end{aligned}$$

Observe $R^{i-j}U_j^{(j)} = U_i^{(j)}$ by Lemma 4.1, so (10) follows. $\square$

Lemma 4.6. *For* $0 \leqslant i \leqslant h$,

$$U_i = \sum_{r=0}^{i} U_i^{(r)} \quad (\textit{direct sum}). \tag{11}$$

Proof. From (10) at $j = 1$,

$$U_i = R^i U_0 + \sum_{r=1}^{i} U_i^{(r)}.$$

Observe $R^i U_0 = R^i U_0^{(0)} = U_i^{(0)}$ by Lemma 4.1, so (11) holds. □

Lemma 4.7. *For* $0 \leqslant i \leqslant d$,

$$U_i = \sum_{r=0}^{m} U_i^{(r)} \quad (\textit{direct sum}), \tag{12}$$

where $m = \min\{i, h, d - i\}$.

Proof. If $0 \leqslant i \leqslant h$, then $m = i$ and (12) follows from Lemma 4.6. Suppose $h < i \leqslant d - h$, and observe $m = h$. From Lemma 4.6,

$$U_h = \sum_{r=0}^{h} U_h^{(r)} \quad (\text{direct sum}).$$

Observe $U_i = R^{i-j} U_h$ by Lemma 4.1, so

$$U_i = R^{i-h} U_h = \sum_{r=0}^{h} R^{i-h} U_h^{(r)} = \sum_{r=0}^{h} U_i^{(r)}.$$

Suppose $d - h < i \leqslant d$. We have $0 \leqslant d - i < h$ and $m = d - i$. From Lemma 4.6,

$$U_{d-i} = \sum_{r=0}^{d-i} U_{d-i}^{(r)}. \tag{13}$$

Apply R^{2i-d} to (13) and use Lemma 4.1 to get (12). □

Theorem 4.8. *V is decomposed as*

$$V = \sum_{r=0}^{h} \sum_{i=r}^{d-r} U_i^{(r)} \quad (\textit{direct sum}). \tag{14}$$

Proof. Follows from (2), Theorem 3.3 and Lemma 4.7. □

5. The subspace $U^{(r)}$

For $0 \leqslant r \leqslant h$, we set

$$U^{(r)} = \sum_{i=r}^{d-r} U_i^{(r)}. \tag{15}$$

Lemma 5.1. *V is decomposed as*

$$V = \sum_{r=0}^{h} U^{(r)} \quad (\textit{direct sum}). \tag{16}$$

Proof. Follows from (15) and Theorem 4.8. □

Lemma 5.2. *For $0 \leqslant r \leqslant h$ and $0 \leqslant i \leqslant d$,*

$$U^{(r)} \cap U_i = \begin{cases} U_i^{(r)} & \textit{if } r \leqslant i \leqslant d-r, \\ 0 & \textit{otherwise}. \end{cases} \tag{17}$$

Proof. Follows from (15) and Theorem 4.8. □

Lemma 5.3. *For $0 \leqslant r \leqslant h$,*

$$RU^{(r)} \subseteq U^{(r)}. \tag{18}$$

Proof. Follows from Lemma 4.1 and (15). □

Lemma 5.4. *For $0 \leqslant r \leqslant h-1$,*

$$LU^{(r)} \subseteq \sum_{s=0}^{r+1} U^{(s)}. \tag{19}$$

Proof. When $r = h-1$, the right side becomes V by (16), so we may assume $r \leqslant h-2$. We show

$$LU_i^{(r)} \subseteq \sum_{s=0}^{r+1} U^{(s)} \quad (r \leqslant i \leqslant d-r) \tag{20}$$

by induction on i. We may assume $i \geqslant 1$, since $LU_0^{(0)} = LU_0 = 0$. When $r \leqslant i \leqslant r+2$, using Lemma 4.6,

$$LU_i^{(r)} \subseteq U_{i-1} = \sum_{s=0}^{i-1} U_{i-1}^{(s)} \subseteq \sum_{s=0}^{i-1} U^{(s)} \subseteq \sum_{s=0}^{r+1} U^{(s)},$$

so (20) holds for $r \leqslant i \leqslant r+2$. Now we assume $r+3 \leqslant i \leqslant d-r$. Let W denote the right side of (20). Pick any vector v in $U_i^{(r)}$. There exists a vector u in $U_{i-3}^{(r)}$

such that $R^3u = v$ by Lemma 4.1. Observe that $Ru \in U^{(r)}_{i-2}$ and $R^2u \in U^{(r)}_{i-1}$, so that each of Lu, $L(Ru)$, $L(R^2u)$ belongs to W by induction. Therefore each of R^3Lu, R^2LRu, RLR^2u belongs to W by Lemma 5.3. Applying (6) to u,

$$R^3Lu - (\beta+1)R^2LRu + (\beta+1)RLR^2u - LR^3u + (\beta+1)\varepsilon_{i-3}R^2u = 0.$$

This implies $Lv = LR^3u \in W$, so (20) holds. Now use (15) to get (19). □

Lemma 5.5. *For* $1 \leqslant r \leqslant h$,

$$LU^{(r)} \subseteq \sum_{s=r-1}^{h} U^{(s)}. \tag{21}$$

Proof. Let W denote the right side of (21). We may assume $r \geqslant 2$, since $W = V$ when $r = 1$. We show

$$LU^{(r)}_{d-i} \subseteq W \quad (r \leqslant i \leqslant d-r) \tag{22}$$

by induction on i.

First we consider the case $i = r$. Pick any vector u in $U^{(r)}_{d-r}$, and apply (6) to u:

$$R^3Lu - (\beta+1)R^2LRu + (\beta+1)RLR^2u - LR^3u + \varepsilon_{d-r}R^2u = 0.$$

Since $Ru = 0$ by Lemma 4.1, this implies $R^3Lu = 0$. On the other hand, using Lemma 4.7,

$$Lu \in U_{d-r-1} = \sum_{s=0}^{m} U^{(s)}_{d-r-1} \quad \text{(direct sum)},$$

where $m = \min\{d-r-1, h, r+1\}$, so that Lu is written as

$$Lu = \sum_{s=0}^{m} u^{(s)}$$

for some vectors $u^{(s)}$ in $U^{(s)}_{d-r-1}$ $(0 \leqslant s \leqslant m)$. Observe that $R^3U^{(s)}_{d-s-2} = 0$ by Lemma 4.1, so that $R^3U^{(s)}_{d-r-1} = 0$ if $r-1 \leqslant s$. Hence we get

$$R^3u^{(s)} = 0 \quad (r-1 \leqslant s \leqslant m),$$

and hence

$$0 = R^3Lu = \sum_{s=0}^{r-2} R^3u^{(s)}.$$

This implies $R^3u^{(s)} = 0$ for $0 \leqslant s \leqslant r-2$. Since

$$R^3|_{U^{(s)}_{d-r-1}} : U^{(s)}_{d-r-1} \to U^{(s)}_{d-r+2}$$

is a bijection for $0 \leqslant s \leqslant r-2$ by Lemma 4.1, we obtain $u^{(s)} = 0$ for $0 \leqslant s \leqslant r-2$. Thus

$$Lu = \sum_{s=r-1}^{m} u^{(s)} \in W.$$

Thus (22) holds at $i = r$.

Now suppose $r + 1 \leqslant i \leqslant d - r$, and pick any vector u in $U_{d-i}^{(r)}$. By induction, each of $L(Ru)$, $L(R^2u)$, $L(R^3u)$ is contained in W, where we remark that $R^2u = 0$ if $i = r + 1$ and $R^3u = 0$ if $i = r + 2$ by Lemma 4.1. Also R^2u is contained in $R^2U^{(r)} \subseteq U^{(r)} \subseteq W$. Applying (6) to u,

$$R^3Lu - (\beta + 1)R^2LRu + (\beta + 1)RLR^2u - LR^3u + (\beta + 1)\varepsilon_{d-i}R^2 = 0,$$

so that $R^3(Lu)$ is contained in W. On the other hand, using Lemma 4.7,

$$Lu \in U_{d-i-1} = \sum_{s=0}^{m} U_{d-i-1}^{(s)} \quad \text{(direct sum)},$$

where $m = \min\{h, i + 1, d - i - 1\}$, so that Lu is written as

$$Lu = \sum_{s=0}^{m} u^{(s)}$$

for some vectors $u^{(s)}$ in $U_{d-i-1}^{(s)}$ $(0 \leqslant s \leqslant m)$. We would like to show $u^{(s)} = 0$ for $0 \leqslant s \leqslant r - 2$. Suppose $u^{(s)} \neq 0$ for some s with $0 \leqslant s \leqslant r - 2$. Observe that $d - i + 2 \leqslant d - s$ and $R^3u^{(s)}$ is contained in $U_{d-i+2}^{(s)}$. This implies $R^3u^{(s)} \neq 0$ since

$$R^3|_{U_{d-i-1}^{(s)}} : U_{d-i-1}^{(s)} \to U_{d-i+2}^{(s)}$$

is a bijection by Lemma 4.1. This contradicts $R^3Lu \in W$. Thus $u^{(s)} = 0$ for $0 \leqslant s \leqslant r - 2$. Therefore

$$Lu = \sum_{s=r-1}^{m} u^{(s)} \in W. \qquad \square$$

Theorem 5.6. *For* $0 \leqslant r \leqslant h$,

$$LU^{(r)} \subseteq U^{(r-1)} + U^{(r)} + U^{(r+1)}, \tag{23}$$

where we set $U^{(-1)} = U^{(h+1)} = 0$.

Proof. Follows from (19) and (21). $\square$

6. The maps $L^{(-)}, L^{(0)}, L^{(+)}$

Let

$$F^{(r)} : V \to U^{(r)} \quad (0 \leqslant r \leqslant h)$$

denote the projection with respect to the direct sum $V = \sum_{r=0}^{h} U^{(r)}$. Observe that for $0 \leqslant r \leqslant h$ and $0 \leqslant s \leqslant h$,

$$F^{(0)} + F^{(1)} + \cdots + F^{(h)} = I, \quad F^{(r)} F^{(r)} = F^{(r)}, \quad F^{(r)} F^{(s)} = 0 \text{ if } r \neq s. \tag{24}$$

We set

$$F_i^{(r)} = F_i F^{(r)} \quad (0 \leqslant r \leqslant h, \ \ 0 \leqslant i \leqslant d).$$

Lemma 6.1. *For* $0 \leqslant r \leqslant h$ *and* $0 \leqslant i \leqslant d$,

(i) $F_i^{(r)} = F^{(r)} F_i = F_i F^{(r)}$,
(ii) $F_0^{(0)} = F_0$ *and* $F_d^{(0)} = F_d$,
(iii) $F_i^{(r)} \neq 0$ *if and only if* $r \leqslant i \leqslant d - r$.

Proof. (i) is clear. (ii) follows from Lemma 4.1. (iii) follows from Lemma 5.2. □

Lemma 6.2. *For* $0 \leqslant r \leqslant h$ *and* $r \leqslant i \leqslant d - r$, $F_i^{(r)} V = U_i^{(r)}$, *and*

$$F_i^{(r)} : V \to U_i^{(r)}$$

is the projection with respect to the direct sum $V = \sum_{r=0}^{h} \sum_{i=r}^{d-r} U_i^{(r)}$.

Proof. Follows from Theorem 4.8, (15) and (16). □

Lemma 6.3. *For* $0 \leqslant r \leqslant h$,

(i) $F^{(r)} R = R F^{(r)}$,
(ii) $F_i^{(r)} R = R F_{i-1}^{(r)}$ $(1 \leqslant i \leqslant d)$,
(iii) $R F_r^{(r)} = 0$.

Proof. (i) follows from Lemma 5.3. (ii) follows from Lemma 2.5. (iii) follows from (ii) and Lemma 6.1(iii). □

Lemma 6.4. *For* $0 \leqslant r \leqslant h$ *and* $0 \leqslant s \leqslant h$, $F^{(r)} L F^{(s)} = 0$ *if* $|r - s| > 1$.

Proof. Follows from (23). □

We set

$$L^{(-)} = \sum_{r=1}^{h} F^{(r-1)} L F^{(r)}, \quad L^{(0)} = \sum_{r=0}^{h} F^{(r)} L F^{(r)}, \quad L^{(+)} = \sum_{r=0}^{h-1} F^{(r+1)} L F^{(r)}. \tag{25}$$

Lemma 6.5

$$L = L^{(-)} + L^{(0)} + L^{(+)}. \tag{26}$$

Proof. Using (24) and Lemma 6.4,

$$\begin{aligned} L = ILI &= \left(\sum_{s=0}^{h} F^{(s)}\right) L \left(\sum_{r=0}^{h} F^{(r)}\right) = \sum_{s=0}^{h}\sum_{r=0}^{h} F^{(s)} L F^{(r)} \\ &= \sum_{r=1}^{h} F^{(r-1)} L F^{(r)} + \sum_{r=0}^{h} F^{(r)} L F^{(r)} + \sum_{r=0}^{h-1} F^{(r+1)} L F^{(r)} \\ &= L^{(-)} + L^{(0)} + L^{(+)}. \qquad \square \end{aligned}$$

Lemma 6.6. *The following hold.*

(i) $F^{(r-1)} L F^{(r)} = L^{(-)} F^{(r)}$ $(1 \leqslant r \leqslant h)$,
(ii) $F^{(r)} L F^{(r)} = L^{(0)} F^{(r)}$ $(0 \leqslant r \leqslant h)$,
(iii) $F^{(r+1)} L F^{(r)} = L^{(+)} F^{(r)}$ $(0 \leqslant r \leqslant h-1)$.

Proof. Using (24),

$$L^{(-)} F^{(r)} = \sum_{s=1}^{h} F^{(s-1)} L F^{(s)} F^{(r)} = F^{(r-1)} L F^{(r)},$$

so (i) holds. We get (ii), (iii) in a similar way. $\square$

Lemma 6.7. *The following hold for* $0 \leqslant r \leqslant h$.

(i) $L^{(0)} F_r^{(r)} = 0$.
(ii) $L^{(+)} F_r^{(r)} = L^{(+)} F_{r+1}^{(r)} = 0$.

Proof. When $r = 0$, $L^{(0)} F_0^{(0)} = L^{(0)} F_0 F^{(0)} = 0$, so we may assume $r \geqslant 1$. Using Lemmas 6.6, 2.5 and 6.1,

$$\begin{aligned} L^{(0)} F_r^{(r)} &= F^{(r)} L F^{(r)} F_r = F^{(r)} L F_r F^{(r)} \\ &= F^{(r)} F_{r-1} L F^{(r)} = F_{r-1}^{(r)} L F^{(r)} = 0, \end{aligned}$$

so (i) holds. The proof of (ii) is similar. $\square$

Theorem 6.8. *The following relations hold.*

$$R^3 L^{(+)} - (\beta+1) R^2 L^{(+)} R + (\beta+1) R L^{(+)} R^2 - L^{(+)} R^3 = 0, \tag{27}$$
$$R^3 L^{(-)} - (\beta+1) R^2 L^{(-)} R + (\beta+1) R L^{(-)} R^2 - L^{(-)} R^3 = 0. \tag{28}$$

Proof. Fix r with $0 \leqslant r \leqslant h-1$. From (6), for $0 \leqslant i \leqslant d-2$,

$$F^{(r+1)}\big(R^3L-(\beta+1)R^2LR+(\beta+1)RLR^2-LR^3+(\beta+1)\varepsilon_i R^2\big)F_iF^{(r)} = 0.$$

Using Lemmas 6.1 and 6.3, this becomes

$$\big(R^3F^{(r+1)}LF^{(r)}-(\beta+1)R^2F^{(r+1)}LF^{(r)}R+(\beta+1)RF^{(r+1)}LF^{(r)} - F^{(r+1)}LF^{(r)}R^3+(\beta+1)\varepsilon_i R^2F^{(r+1)}F^{(r)}\big)F_i=0.$$

We have $F^{(r+1)}F^{(r)}=0$ by (24), and $F^{(r+1)}LF^{(r)}=L^{(+)}F^{(r)}$ by Lemma 6.6. So

$$\big(R^3L^{(+)}-(\beta+1)R^2L^{(+)}R+(\beta+1)RL^{(+)}R^2-L^{(+)}R^3\big)F^{(r)}F_i=0 \tag{29}$$

holds for $0 \leqslant r \leqslant h-1$, $0 \leqslant i \leqslant d-2$. Observe that (29) also holds for $r=h$, and for $i=d-1,d$. Thus (29) holds for all r,i with $0 \leqslant r \leqslant h$, $0 \leqslant i \leqslant d$. This implies (27) by Lemma 6.2. The proof of (28) is similar. □

Lemma 6.9. *For* $0 \leqslant r \leqslant h$ *and* $0 \leqslant i \leqslant d-2$,

$$\big(R^3L^{(0)}-(\beta+1)R^2L^{(0)}R+(\beta+1)RL^{(0)}R^2-L^{(0)}R^3+(\beta+1)\varepsilon_i R^2\big)F_i^{(r)} = 0. \tag{30}$$

Proof. Similar to the proof of (27). □

7. Some remarks on q-integers

For the rest of this paper, we fix a scalar q taken from the algebraic closure $\overline{\mathbb{K}}$ such that

$$\beta = q^2+q^{-2}.$$

We assume q is not a root of unity.

For a nonnegative integer n, we use the notation of q-integers;

$$[n] = \frac{q^n-q^{-n}}{q-q^{-1}},$$

$$[n]! = \prod_{k=1}^{n}[k].$$

We also use the notation of q-binomial coefficients;

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{[n]!}{[i]![n-i]!} \quad (0 \leqslant i \leqslant n).$$

Observe that

$$[0] = 0, \quad [1] = 1, \quad [0]! = 1, \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1, \quad \begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} n \\ n-1 \end{bmatrix} = [n].$$

Also observe that the q-integers and the q-binomial coefficients are nonzero if $n \neq 0$ since q is not a root of unity.

We shall use the following formulas, each of which can be easily verified:

$$[n+1] + [n-1] = [2][n] \quad (n \geqslant 1), \tag{31}$$

$$[3]\begin{bmatrix} n-1 \\ 2 \end{bmatrix} - \begin{bmatrix} n-2 \\ 2 \end{bmatrix} = [n][n-2] \quad (n \geqslant 4), \tag{32}$$

$$[3]\begin{bmatrix} n \\ 2 \end{bmatrix} - [n][n-2] = \begin{bmatrix} n+1 \\ 2 \end{bmatrix} \quad (n \geqslant 2). \tag{33}$$

In particular,

$$[3][3] - 1 = [4][2], \tag{34}$$

$$[3] - 1 = \frac{[4]}{[2]}. \tag{35}$$

Lemma 7.1. *The following hold.*

(i) $\beta + 1 = [3] \in \mathbb{K}$.
(ii) $[2n-1] \in \mathbb{K}$ $(n \geqslant 1)$.
(iii) $[2][2n] \in \mathbb{K}$, $[2n]/[2] \in \mathbb{K}$ $(n \geqslant 0)$.
(iv) *If* $[2] = q + q^{-1} \in \mathbb{K}$, *then* $[2n] \in \mathbb{K}$ $(n \geqslant 0)$.
(v) $\begin{bmatrix} n \\ 2 \end{bmatrix} \in \mathbb{K}$ $(n \geqslant 2)$.

Proof. Each can be easily verified. $\square$

8. A relation between R and $L^{(+)}$

In this section, we fix an integer r such that $0 \leqslant r \leqslant h-1$, and set

$$X_i = L^{(+)} F_i^{(r)} \quad (0 \leqslant i \leqslant d).$$

Lemma 8.1. *For* $r \leqslant i \leqslant d - r - 3$,

$$R^3 X_i - [3]R^2 X_{i+1} R + [3] R X_{i+2} R^2 - X_{i+3} R^3 = 0. \tag{36}$$

Proof. From (27),

$$\big(R^3L^{(+)} - [3]R^2L^{(+)}R + [3]RL^{(+)}R^2 - L^{(+)}R^3\big)F_i^{(r)} = 0.$$

Using Lemma 6.3, this becomes

$$R^3L^{(+)}F_i^{(r)} - [3]R^2L^{(+)}F_{i+1}^{(r)}R + [3]RL^{(+)}F_{i+2}^{(r)}R^2 - L^{(+)}F_{i+3}^{(r)}R^3 = 0.$$

□

Lemma 8.2. $X_r = X_{r+1} = 0.$

Proof. Follows from Lemma 6.7. □

Lemma 8.3. *For* $r + 2 \leqslant i \leqslant d - r - 1,$

$$\begin{bmatrix} i-r+1 \\ 2 \end{bmatrix} RX_i - \begin{bmatrix} i-r \\ 2 \end{bmatrix} X_{i+1}R = 0. \tag{37}$$

Proof. We show (37) by induction. Setting $i = r$ in (36), and using Lemma 8.2,

$$0 = [3]RX_{r+2}R^2 - X_{r+3}R^3 = ([3]RX_{r+2} - X_{r+3}R)R^2. \tag{38}$$

Observe that the restriction

$$R^2|_{U_r^{(r)}} : U_r^{(r)} \to U_{r+2}^{(r)}$$

is a bijection by Lemma 4.1, we may remove the factor R^2 in (38) to get

$$[3]RX_{r+2} - X_{r+3}R = 0. \tag{39}$$

so that (37) holds at $i = r + 2$.

Setting $i = r + 1$ in (36), and using Lemma 8.2,

$$-[3]R^2X_{r+2}R + [3]RX_{r+3}R^2 - X_{r+4}R^3 = 0.$$

Removing the factor R,

$$-[3]R^2X_{r+2} + [3]RX_{r+3}R - X_{r+4}R^2 = 0. \tag{40}$$

Multiplying (39) by R from the left,

$$[3]R^2X_{r+2}R - X_{r+3}R^2 = 0. \tag{41}$$

Adding (40) and (41),

$$([3] - 1)RX_{r+3}R - X_{r+4}R^2 = 0.$$

Use (35), multiply [3], and remove R to get

$$\frac{[4][3]}{[2]}RX_{r+3} - [3]X_{r+4}R = 0. \tag{42}$$

Thus (37) holds at $i = r + 3$.

Now suppose $r+4 \leqslant i \leqslant d-r-1$, and (37) holds at $i-1$, $i-2$;

$$\begin{bmatrix} i-r-1 \\ 2 \end{bmatrix} RX_{i-2} - \begin{bmatrix} i-r-2 \\ 2 \end{bmatrix} X_{i-1}R = 0, \tag{43}$$

$$\begin{bmatrix} i-r \\ 2 \end{bmatrix} RX_{i-1} - \begin{bmatrix} i-r-1 \\ 2 \end{bmatrix} X_i R = 0. \tag{44}$$

From (36),

$$R^3X_{i-2} - [3]R^2X_{i-1}R + [3]RX_iR^2 - X_{i+1}R^3 = 0. \tag{45}$$

Multiplying (43) by $(-1)R^2$,

$$-\begin{bmatrix} i-r-1 \\ 2 \end{bmatrix} R^3X_{i-2} + \begin{bmatrix} i-r-2 \\ 2 \end{bmatrix} R^2X_{i-1}R = 0. \tag{46}$$

Multiply (45) by $\begin{bmatrix} i-r-1 \\ 2 \end{bmatrix}$, and add (46) to get

$$\left(-[3]\begin{bmatrix} i-r-1 \\ 2 \end{bmatrix} + \begin{bmatrix} i-r-2 \\ 2 \end{bmatrix}\right) R^2X_{i-1}R + [3]\begin{bmatrix} i-r-1 \\ 2 \end{bmatrix} RX_iR^2 - \begin{bmatrix} i-r-1 \\ 2 \end{bmatrix} X_{i+1}R^3 = 0.$$

Using (32) and removing R,

$$-[i-r][i-r-2]R^2X_{i-1} + [3]\begin{bmatrix} i-r-1 \\ 2 \end{bmatrix} RX_iR - \begin{bmatrix} i-r-1 \\ 2 \end{bmatrix} X_{i+1}R^2 = 0. \tag{47}$$

Multiplying (44) by R,

$$\begin{bmatrix} i-r \\ 2 \end{bmatrix} R^2X_{i-1} - \begin{bmatrix} i-r-1 \\ 2 \end{bmatrix} RX_iR = 0. \tag{48}$$

Eliminating the term R^2X_{i-1} from (47), (48), and dividing by $\begin{bmatrix} i-r-1 \\ 2 \end{bmatrix}$,

$$\left([3]\begin{bmatrix} i-r \\ 2 \end{bmatrix} - [i-r][i-r-2]\right) RX_iR - \begin{bmatrix} i-r \\ 2 \end{bmatrix} X_{i+1}R^2 = 0.$$

Now use (33) and remove R to get (37). □

Theorem 8.4. *For $0 \leqslant r \leqslant h-1$ and $r+2 \leqslant i \leqslant d-r-1$,*

$$RL^{(+)} - \frac{[i-r-1]}{[i-r+1]} L^{(+)}R$$

vanishes on $U_i^{(r)}$.

Proof. Follows from (37) and the definition of X_i. □

9. A relation between R and $L^{(-)}$

In this section, we fix an integer r such that $1 \leqslant r \leqslant h$, and set

$$Y_i = L^{(-)} F_i^{(r)} \quad (0 \leqslant i \leqslant d).$$

We set $Y_{d+1} = 0$.

Lemma 9.1

$$Y_{d-r+1} = Y_{d-r+2} = 0.$$

Proof. Follows from Lemma 6.1. □

Lemma 9.2. *For $r \leqslant i \leqslant d - r - 1$,*

$$R^3 Y_i - [3] R^2 Y_{i+1} R + [3] R Y_{i+2} R^2 - Y_{i+3} R^3 = 0. \tag{49}$$

Proof. Similar to the proof of (36). □

Lemma 9.3. *For $r \leqslant i \leqslant d - r - 1$,*

$$\begin{bmatrix} d-r-i+1 \\ 2 \end{bmatrix} R Y_i - \begin{bmatrix} d-r-i+2 \\ 2 \end{bmatrix} Y_{i+1} R = 0. \tag{50}$$

Proof. Using Lemma 9.1, (49) at $i = d - r - 1$ becomes

$$R^3 Y_{d-r-1} - [3] R^2 Y_{d-r} R = 0.$$

Removing R^2,

$$R Y_{d-r-1} - [3] Y_{d-r} R = 0. \tag{51}$$

Thus (50) holds at $i = d - r - 1$.

Using Lemma 9.1, (49) at $i = d - r - 2$ becomes

$$R^3 Y_{d-r-2} - [3] R^2 Y_{d-r-1} R + [3] R Y_{d-r} R^2 = 0.$$

Removing R,

$$R^2 Y_{d-r-2} - [3] R Y_{d-r-1} R + [3] Y_{d-r} R^2 = 0. \tag{52}$$

Multiplying (51) by R from the right,

$$R Y_{d-r-1} R - [3] R_{d-r} R^2 = 0. \tag{53}$$

Adding (52) and (53), and removing R,

$$R Y_{d-r-2} - ([3] - 1) Y_{d-r-1} R = 0.$$

Multiplying by [3] and using (35), this becomes

$$[3] R Y_{d-r-2} - \begin{bmatrix} 4 \\ 2 \end{bmatrix} Y_{d-r-1} R = 0.$$

Thus (50) holds at $i = d - r - 2$.

Suppose $r \leqslant i \leqslant d - r - 3$, and (50) holds at $i + 1, i + 2$;

$$\begin{bmatrix} d-r-i-1 \\ 2 \end{bmatrix} RY_{i+2} - \begin{bmatrix} d-r-i \\ 2 \end{bmatrix} Y_{i+3}R = 0, \tag{54}$$

$$\begin{bmatrix} d-r-i \\ 2 \end{bmatrix} RY_{i+1} - \begin{bmatrix} d-r-i+1 \\ 2 \end{bmatrix} Y_{i+2}R = 0. \tag{55}$$

From (49)

$$R^3Y_i - [3]R^2Y_{i+1}R + [3]RY_{i+2}R^2 - Y_{i+3}R^3 = 0. \tag{56}$$

Multiplying (54) by $(-1)R^2$,

$$-\begin{bmatrix} d-r-i-1 \\ 2 \end{bmatrix} RY_{i+2}R^2 + \begin{bmatrix} d-r-i \\ 2 \end{bmatrix} Y_{i+3}R^3 = 0. \tag{57}$$

Eliminating the term of $Y_{i+3}R^3$ from (56) and (57),

$$\begin{aligned} &\begin{bmatrix} d-r-i \\ 2 \end{bmatrix} R^3Y_i - [3]\begin{bmatrix} d-r-i \\ 2 \end{bmatrix} R^2Y_{i+1}R \\ &\quad + \left([3]\begin{bmatrix} d-r-i \\ 2 \end{bmatrix} - \begin{bmatrix} d-r-i-1 \\ 2 \end{bmatrix}\right) RY_{i+2}R^2 = 0. \end{aligned} \tag{58}$$

Using (32) and removing R,

$$\begin{aligned} &\begin{bmatrix} d-r-i \\ 2 \end{bmatrix} R^2Y_i - [3]\begin{bmatrix} d-r-i \\ 2 \end{bmatrix} RY_{i+1}R \\ &\quad + [d-r-i+1][d-r-i-1]Y_{i+2}R^2 = 0. \end{aligned} \tag{59}$$

Multiplying (55) by R,

$$\begin{bmatrix} d-r-i \\ 2 \end{bmatrix} RY_{i+1}R - \begin{bmatrix} d-r-i+1 \\ 2 \end{bmatrix} Y_{i+2}R^2 = 0. \tag{60}$$

Eliminating the term of $Y_{i+2}R^2$ from (59) and (60), dividing by $\begin{bmatrix} d-r-i \\ 2 \end{bmatrix}$, and removing R,

$$\begin{aligned} &\begin{bmatrix} d-r-i+1 \\ 2 \end{bmatrix} RY_i - \Bigg([3]\begin{bmatrix} d-r-i+1 \\ 2 \end{bmatrix} \\ &\quad - [d-r-i+1][d-r-i-1]\Bigg) Y_{i+1}R = 0. \end{aligned}$$

Now use (33) to get (50). $\square$

Theorem 9.4. *For* $1 \leqslant r \leqslant h$ *and* $r \leqslant i \leqslant d-r-1$,

$$RL^{(-)} - \frac{[d-r-i+2]}{[d-r-i]} L^{(-)} R$$

vanishes on $U_i^{(r)}$.

Proof. Follows from (50) and the definition of Y_i. $\square$

10. A relation between R and $L^{(0)}$

In this section, we fix an integer r such that $0 \leqslant r \leqslant h$, and set

$$Z_i = L^{(0)} F_i^{(r)} \quad (0 \leqslant i \leqslant d).$$

Lemma 10.1. $Z_r = Z_{d-r+1} = 0$.

Proof. Follows from Lemmas 6.1 and 6.7. $\square$

Lemma 10.2. *For* $r \leqslant i \leqslant d-r-2$,

$$R^3 Z_i - [3] R^2 Z_{i+1} R + [3] R Z_{i+2} R^2 - Z_{i+3} R^3 + [3] \varepsilon_i R^2 F_i^{(r)} = 0. \tag{61}$$

Proof. Follows from (30). $\square$

Lemma 10.3. *For* $r+1 \leqslant i \leqslant d-r-1$,

$$-\begin{bmatrix} i-r+2 \\ 2 \end{bmatrix} R^2 Z_i + [i-r][i-r+2] R Z_{i+1} R - \begin{bmatrix} i-r+1 \\ 2 \end{bmatrix} Z_{i+2} R^2$$
$$+ [3] \sum_{j=r}^{i-1} \begin{bmatrix} j-r+2 \\ 2 \end{bmatrix} \varepsilon_j R F_i^{(r)} = 0. \tag{62}$$

Proof. Using Lemma 10.1, (61) at $i = r$ becomes

$$-[3] R^2 Z_{r+1} R + [3] R Z_{r+2} R^2 - Z_{r+3} R^3 + [3] \varepsilon_r R^2 F_r^{(r)} = 0.$$

Using Lemma 6.3 and removing R,

$$-[3] R^2 Z_{r+1} + [3] R Z_{r+2} R - Z_{r+3} R^2 + [3] \varepsilon_r R F_{r+1}^{(r)} = 0. \tag{63}$$

So (62) holds at $i = r+1$.

Suppose $r+2 \leqslant i \leqslant d-r-1$. Multiplying (62) at $i-1$ by R,

$$-\begin{bmatrix} i-r+1 \\ 2 \end{bmatrix} R^3 Z_{i-1} + [i-r-1][i-r+1] R^2 Z_i R - \begin{bmatrix} i-r \\ 2 \end{bmatrix} R Z_{i+1} R^2$$
$$+ [3] \sum_{j=r}^{i-2} \begin{bmatrix} j-r+2 \\ 2 \end{bmatrix} \varepsilon_j R^2 F_{i-1}^{(r)} = 0. \tag{64}$$

From (61) at $i-1$,

$$R^3 Z_{i-1} - [3]R^2 Z_i R + [3]R Z_{i+1} R^2 - Z_{i+2} R^3 + [3]\varepsilon_{i-1} R^2 F^{(r)}_{i-1} = 0. \quad (65)$$

Eliminating the term of $R^3 Z_{i-1}$ from (64) and (65), using Lemma 6.3, and removing R,

$$\begin{aligned}
&-\left(-[i-r-1][i-r+1] + [3]\begin{bmatrix} i-r+1 \\ 2 \end{bmatrix}\right) R^2 Z_i \\
&\quad + \left([3]\begin{bmatrix} i-r+1 \\ 2 \end{bmatrix} - \begin{bmatrix} i-r \\ 2 \end{bmatrix}\right) R Z_{i+1} R \\
&\quad - \begin{bmatrix} i-r+1 \\ 2 \end{bmatrix} Z_{i+2} R^2 + [3]\sum_{j=r}^{i-1} \begin{bmatrix} j-r+2 \\ 2 \end{bmatrix} \varepsilon_j R F^{(r)}_i = 0.
\end{aligned}$$

Now use (32) and (33) to get (62). □

Lemma 10.4. *For $r \leqslant i \leqslant d-r-2$,*

$$\begin{aligned}
&\begin{bmatrix} d-r-i \\ 2 \end{bmatrix} R^2 Z_i - [d-r-i-1][d-r-i+1] R Z_{i+1} R \\
&\quad + \begin{bmatrix} d-r-i+1 \\ 2 \end{bmatrix} Z_{i+2} R^2 + [3]\sum_{j=i}^{d-r-2} \begin{bmatrix} d-r-j \\ 2 \end{bmatrix} \varepsilon_j R F^{(r)}_i = 0. \quad (66)
\end{aligned}$$

Proof. Using Lemma 10.1, (61) at $i = d-r-2$ becomes

$$R^3 Z_{d-r-2} - [3]R^2 Z_{d-r-1} R + [3]R Z_{d-r} R^2 + [3]\varepsilon_{d-r-2} R^2 F^{(r)}_{d-r-2} = 0.$$

Removing R,

$$R^2 Z_{d-r-2} - [3]R Z_{d-r-1} R + [3] Z_{d-r} R^2 + [3]\varepsilon_{d-r-2} F^{(r)}_{d-r-2} = 0. \quad (67)$$

So (66) holds at $i = d-r-2$.

Suppose $r \leqslant i \leqslant d-r-3$. Multiplying (66) at $i+1$ by R, and using Lemma 6.3,

$$\begin{aligned}
&\begin{bmatrix} d-r-i-1 \\ 2 \end{bmatrix} R^2 Z_{i+1} R - [d-r-i-2][d-r-i] R Z_{i+2} R^2 \\
&\quad + \begin{bmatrix} d-r-i \\ 2 \end{bmatrix} Z_{i+3} R^3 + [3]\sum_{j=i+1}^{d-r-2} \begin{bmatrix} d-r-j \\ 2 \end{bmatrix} \varepsilon_j R^2 F^{(r)}_i = 0. \quad (68)
\end{aligned}$$

Eliminating the term of $Z_{i+3} R^3$ from (68) and (61), and removing R,

$$\begin{bmatrix} d-r-i \\ 2 \end{bmatrix} R^2 Z_i - \left([3]\begin{bmatrix} d-r-i \\ 2 \end{bmatrix} - \begin{bmatrix} d-r-i-1 \\ 2 \end{bmatrix}\right) R Z_{i+1} R$$

$$+\left([3]\begin{bmatrix} d-r-i \\ 2 \end{bmatrix} - [d-r-i-2][d-r-i]\right) Z_{i+2}R^2$$
$$+\left([3]\begin{bmatrix} d-r-i \\ 2 \end{bmatrix}\varepsilon_i + [3]\sum_{j=i+1}^{d-r-2}\begin{bmatrix} d-r-j \\ 2 \end{bmatrix}\varepsilon_j\right) RF_i^{(r)} = 0.$$

Now use (32) and (33) to get (66). □

Lemma 10.5. *For* $r+1 \leqslant i \leqslant d-r-1$,

$$\frac{1}{[2]}[d-2r+1][i-r+1][d-r-i]RZ_i$$
$$-\frac{1}{[2]}[d-2r+1][i-r][d-r-i+1]Z_{i+1}R - [3]e_i F_i^{(r)} = 0, \qquad (69)$$

where

$$e_i = \begin{bmatrix} d-r-i+1 \\ 2 \end{bmatrix}\sum_{j=r}^{i-1}\begin{bmatrix} j-r+2 \\ 2 \end{bmatrix}\varepsilon_j$$
$$+\begin{bmatrix} i-r+1 \\ 2 \end{bmatrix}\sum_{j=i}^{d-r-2}\begin{bmatrix} d-r-j \\ 2 \end{bmatrix}\varepsilon_j. \qquad (70)$$

Proof. Using Lemma 10.1, (62) at $i = d-r-1$ implies (69) at $i = d-r-1$. Suppose $r+1 \leqslant i \leqslant d-r-2$. Eliminating the term of $Z_{i+2}R^2$ from (62) and (66), and removing R,

$$\left(-\begin{bmatrix} i-r+2 \\ 2 \end{bmatrix}\begin{bmatrix} d-r-i+1 \\ 2 \end{bmatrix} + \begin{bmatrix} i-r+1 \\ 2 \end{bmatrix}\begin{bmatrix} d-r-i \\ 2 \end{bmatrix}\right) RZ_i$$
$$+\Bigg([i-r][i-r+2]\begin{bmatrix} d-r-i+1 \\ 2 \end{bmatrix}$$
$$-[d-r-i-1][d-r-i+1]\begin{bmatrix} i-r+1 \\ 2 \end{bmatrix}\Bigg) Z_{i+1}R$$
$$+[3]\Bigg(\begin{bmatrix} d-r-i+1 \\ 2 \end{bmatrix}\sum_{j=r}^{i-1}\begin{bmatrix} j-r+2 \\ 2 \end{bmatrix}\varepsilon_j$$
$$+\begin{bmatrix} i-r+1 \\ 2 \end{bmatrix}\sum_{j=i}^{d-r-2}\begin{bmatrix} d-r-j \\ 2 \end{bmatrix}\varepsilon_j\Bigg) F_i^{(r)}$$
$$= 0. \qquad (71)$$

Now use (32) and (33) to get (69). □

Lemma 10.6 [1, Lemma 8.6 and Theorem 11.1]. *There are scalars* a, a^*, b, b^*, c, c^* *in the algebraic closure* $\overline{\mathbb{K}}$ *such that*

$$\theta_i = aq^{2i} + bq^{-2i} + c \quad (0 \leqslant i \leqslant d), \tag{72}$$
$$\theta_i^* = a^*q^{2i} + b^*q^{-2i} + c^* \quad (0 \leqslant i \leqslant d). \tag{73}$$

We fix scalars a, a^*, b, b^*, c, c^* which satisfy (72) and (73).

Lemma 10.7. *For* $0 \leqslant i \leqslant d-2$,

$$\varepsilon_i = [2](q-q^{-1})^3(aa^*q^{4i+4} - bb^*q^{-(4i+4)}). \tag{74}$$

Proof. Follows from (5), (72) and (73) by a routine computation. □

Lemma 10.8. *For* $r+1 \leqslant i \leqslant d-r-1$, *the scalar* e_i *defined by* (70) *is given by*

$$e_i = \frac{[2]}{[3]}[d-2r+1]\begin{bmatrix} i-r+1 \\ 2 \end{bmatrix}\begin{bmatrix} d-r-i+1 \\ 2 \end{bmatrix}(q-q^{-1})^3 \times (aa^*q^{d+2i} - bb^*q^{-d-2i}). \tag{75}$$

Proof. Follows from (70) and (74) by a long but routine computation. □

Theorem 10.9. *For* $0 \leqslant r \leqslant h$ *and* $r+1 \leqslant i \leqslant d-r-1$, *the following map vanishes on* $U_i^{(r)}$:

$$RL^{(0)} - \frac{[i-r][d-r-i+1]}{[i-r+1][d-r-i]}L^{(0)}R - [i-r][d-r-i+1]\mu_i I,$$

where

$$\mu_i = (q-q^{-1})^3(aa^*q^{d+2i} - bb^*q^{-d-2i}).$$

Proof. Follows from Lemmas 7.1, 10.5, and (75). □

References

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